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The Zeros of the Partial Sums of the Exponential Function*

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1. The zeros of the partial sum

$$S_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!}$$

of e^z tend to infinity as $n \to \infty$. A detailed study of their behavior was made by Szegö [3] who showed, among many other things, that if $z_1^{(n)},..., z_n^{(n)}$ are the zeros of $S_n(z)$, then the point set $\{z_1^{(n)}/n,..., z_n^{(n)}/n\}$ has as its points of accumulation, as $n \to \infty$, the simple closed loop A of the curve defined by $|ze^{1-z}| = 1$. Moreover, if $w = ze^{1-z}$, then $\arg w$ increases monotonely from 0 to 2π as A is traversed from $z = 1$ in the positive direction, and Szegö [3] also showed that if v_n is the number of zeros of $S_n(z)$ in the sector $\theta_1 \leqslant \theta \leqslant \theta_2$, then

$$\lim_{n\to\infty} (v_n/n) = (\varphi_2 - \varphi_1)/2\pi, \tag{1}$$

where if z_1 and z_2 are the points of A with arguments θ_1 and θ_2, respectively, $\varphi_1 = \arg w(z_1)$ and $\varphi_2 = \arg w(z_2)$. (These results were also obtained later independently by Dieudonné [1]. See also Rosenbloom [2] where generalizations are given.)

In view of (1) we may conclude, in particular, that every infinite sector symmetric about the positive real axis contains zeros of $S_n(z)$ for n sufficiently large. (This same conclusion follows from the fact that if there were a sector containing $o(n)$ zeros of $S_n(z)$ as $n \to \infty$, then e^z would be an entire function of order zero (Cf. Rosenbloom [2]). In contrast to this, Varga [4] showed that there exists a constant $B > 0$ such that S_n has no zeros, for $n = 0, 1, 2, \ldots$ in $|\operatorname{Im} z| \leqslant B$, $\operatorname{Re} z \geqslant 0$. Our purpose here is to demonstrate the existence of a "parabolic" domain free of zeros of S_n, for n sufficiently large.

1. An easy computation verifies that

$$S_n(z) = \int_0^\infty \frac{(z+t)^n}{n!} e^{-t}\, dt,$$

and if we put $z = n + w\sqrt{n}$,

$$\frac{S_n(n + w\sqrt{n})}{e^{n+w\sqrt{n}}} = \frac{\sqrt{2\pi n}\, e^{-n} n^n}{n!} \frac{1}{\sqrt{2\pi}} \int_w^\infty \left(1 + \frac{\zeta}{\sqrt{n}}\right)^n e^{-\sqrt{n}\zeta}\, d\zeta, \tag{2}$$

the path of integration in (2) being the horizontal line from w to the right to ∞. Thus, if we put

$$\varphi_n(\zeta) = \left(1 + \frac{\zeta}{\sqrt{n}}\right)^n e^{-\sqrt{n}\zeta},$$

then $w = u + iv$ is a zero of

$$\Phi_n(w) = \int_w^\infty \varphi_n(\zeta)\, d\zeta$$

if, and only if, $z = x + iy = n + (u + iv)\sqrt{n}$ is a zero of $S_n(z)$; that is, if

$$x = n + u\sqrt{n}; \qquad y = v\sqrt{n}. \tag{3}$$

If Φ_n has a real zero, n must be odd and $x = n + u\sqrt{n}$ is the unique real zero of $S_n(z)$. (Pólya and Szegö, Vol. I, p. 81). For each zero $u + iv$ of Φ_n with $v \neq 0$, a distinct parabola

$$x = (y/v)^2 + u(y/v) \tag{4}$$

is defined, which contains the corresponding zero $x + iy$ of $S_n(z)$. Our interest is in the limit points of the zeros of Φ_n as $n \to \infty$.

LEMMA 1. *For each complex number ζ,*

$$\lim_{n\to\infty} \varphi_n(\zeta) = e^{-\zeta^2/2},$$

and the convergence is uniform on every compact set in the ζ-plane.

Proof. Straightforward after taking logarithms.

LEMMA 2.

$$\lim_{n\to\infty} \Phi_n(a) = \int_a^\infty e^{-t^2/2}\, dt, \qquad a \geqslant 2. \tag{5}$$

Proof. The lemma will follow from the dominated convergence theorem and Lemma 1 when we show

$$\varphi_n(t) \leqslant e^{4/3} e^{-t}; \qquad t \geqslant 2, \quad n \geqslant 5. \tag{6}$$

But if we write $m = \sqrt{n}$ for convenience,

$$f(t) = \log e^t \varphi_n(t) = m^2 \log\left(1 + \frac{t}{m}\right) + (1 - m)t,$$

and if $m > 2, f'(t) < 0$ if $t \geqslant 2$. Thus, if $t \geqslant 2$ and $m > 2$,

$$f(t) \leqslant f(2) = 4 \sum_{k=3}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{2}{m}\right)^{k-2} \leqslant \frac{8}{3m} \leqslant \frac{4}{3}$$

and (6) is established.

LEMMA 3. *If* $w_0 = u_0 + iv_0$, $(v_0 \geqslant 0)$,

$$\Phi_n(w_0) = \Phi_n(u_0) - i \int_0^{v_0} \varphi_n(u_0 + iv)\, dv.$$

Proof. Φ_n is an entire function by definition, and

$$\lim_{R\to\infty} \int_0^{v_0} \varphi_n(R + iv)\, dv = 0.$$

Lemmas 1–3 imply

THEOREM 1. *$\Phi_n(w)$ converges uniformly to*

$$F(w) = \int_w^\infty e^{-\zeta^2/2}\, d\zeta \tag{7}$$

on any compact set in $\operatorname{Im} w \geqslant 0$.

Remark.

$$F(w) = \sqrt{\frac{\pi}{2}}\, \operatorname{erfc}\left(\frac{\sqrt{2}}{2} w\right).$$

We have seen that for each n the zeros of $S_n(z)$ in the upper half-plane (and for reasons of symmetry we need only consider these) lie on parabolas described by (4). Hence, in view of Theorem 1 and Stirling's formula, for n sufficiently large the zeros in the upper half-plane, $x + iy$, of $S_n(z)$ lie arbitrarily close to the parabolas defined by

$$x = (y/v)^2 + u(y/v), \tag{8}$$

where now $u + iv$ is any zero of $F(w)$ with $v > 0$. Therefore, we turn next to a study of the zeros of $F(w)$.

2. $F(w)$ is an entire function of order 2 since its derivative is. $F(w)$ has infinitely many zeros, for if it has a finite number of zeros, then $F(w)/(w - w_1) \cdots (w - w_k)$ is entire, of order 2, and free of zeros so that by the Hadamard factorization theorem

$$F(w) = (w - w_1) \cdots (w - w_k)\, e^{az^2+bz+c}, \qquad a \neq 0.$$

But

$$\lim_{t\to-\infty} F(t) > 0,$$

while

$$\lim_{t\to-\infty} (t - w_1) \cdots (t - w_k)\, e^{at^2+bt+c}$$

is either zero or ∞.

The power series expansion of F about the origin has real coefficients, so its zeros are complex conjugates (F has no real zeros). We restrict our attention to the upper half-plane.

THEOREM 2. *With $w = u + iv$, $F(w)$ has no zeros in $v > 0$, $uv \geqslant -\pi$.*

Proof.

$$F(w) = \int_w^\infty e^{-(\zeta+iv)^2/2}\, d\zeta = e^{v^2/2} \int_u^\infty e^{-(\xi^2/2)}\, e^{-i\xi v}\, d\xi$$

$$= e^{(v^2-u^2)/2}\, e^{-iuv} \int_0^\infty e^{-(t^2/2)-tu}\, e^{-itv}\, dt.$$

We put

$$K(u, v) = -\mathrm{Im} \int_0^\infty e^{-(t^2/2)-tu}\, e^{-itv}\, dt = \int_0^\infty e^{-(t^2/2)-tu} \sin tv\, dt,$$

and complete our proof by showing that $K(u, v) \neq 0$ for $v > 0$, $uv \geqslant -\pi$.

Let

$$A(j) = \int_{j\pi/v}^{[(j+1)\pi]/v} e^{-(t^2/2)-tu} \sin tv \, dt$$

so that

$$K(u, v) = \sum_{j=0}^{\infty} A(j).$$

Put $s = tv - j\pi$; then

$$A(j) = \frac{(-1)^j}{v} \int_0^{\pi} e^{-((s+j\pi)/v)^2/2-u(s+j\pi)/v} \sin s \, ds.$$

If $t \geqslant (2\pi)/v$ and $uv \geqslant -\pi$,

$$h(t) = e^{-(t^2/2)-tu}$$

is strictly monotone decreasing. Hence, $|A(j)|$, $j = 2, 3, \ldots$ is strictly monotone decreasing and $\operatorname{sgn} A(j) = (-1)^j$, $j = 2, 3, \ldots$. Therefore,

$$\sum_{j=2}^{\infty} A(j) > 0.$$

Now

$$A(0) = \frac{1}{v} \int_0^{\pi} e^{-(s^2/2v^2)-(u/v)s} \sin s \, ds = \frac{1}{v} \int_0^{\pi} e^{-(\pi-s)^2/2v^2-(u/v)(\pi-s)} \sin s \, ds$$

and

$$A(1) = -\frac{1}{v} \int_0^{\pi} e^{-(\pi+s)^2/2v^2-(u/v)(\pi+s)} \sin s \, ds$$

so that

$$A(0) + A(1) = \frac{1}{v} \int_0^{\pi} \left[h\left(\frac{\pi - s}{v}\right) - h\left(\frac{\pi + s}{v}\right)\right] \sin s \, ds.$$

Moreover,

$$h\left(\frac{\pi - s}{v}\right) \geqslant h\left(\frac{\pi + s}{v}\right)$$

if, and only if

$$\frac{(\pi - s)^2}{2v^2} + \frac{(\pi - s)u}{v} \leqslant \frac{(\pi + s)^2}{2v^2} + \frac{(\pi + s)u}{v},$$

if, and only if,

$$\frac{\pi}{v} + u \geqslant 0,$$

which holds by hypothesis. Therefore, $A(0) + A(1) \geqslant 0$ and $K(u, v) > 0$.

THEOREM 3. *There exists a positive constant, c_0, such that $F(w)$ has no zero, $u + iv$, satisfying $v > 0$ and $u + v + c_0 \leqslant 0$. (Indeed,* $\operatorname{Re} F(w)$ *has no such zero.)*

Proof. Suppose $v > 0$. Replacing paths as in Lemma 3 we obtain

$$F(w) = -i \int_0^v e^{-(u^2-\eta^2)/2} e^{-iu\eta}\, d\eta + \int_u^\infty e^{-\xi^2/2}\, d\xi,$$

hence,

$$\begin{aligned} \operatorname{Re} F(w) &= \int_u^\infty e^{-\xi^2/2}\, d\xi - \int_0^v e^{(\eta^2-u^2)/2} \sin u\eta\, d\eta \\ &> \frac{\sqrt{2\pi}}{2} - \int_0^v e^{(\eta^2-u^2)/2} \sin u\eta\, d\eta. \end{aligned} \tag{9}$$

Suppose $u + v = -c$, $c > 0$.

$$\begin{aligned} R = \left| \int_0^v e^{(\eta^2-u^2)/2} \sin u\eta\, d\eta \right| &< ve^{(v^2-u^2)/2} = ve^{[(v-u)/2](v-u} \\ &\leqslant -(u + c)\, e^{c(c+2u)/2} = e^{c^2/2}[-(u + c)\, e^{uc}]. \end{aligned}$$

Now $-(u + c)\, e^{uc}$ is positive for $u < -c$ and assumes its maximum at $u = -(c + c^{-1})$ so that

$$R \leqslant \frac{e^{-c^2/2}}{ec}.$$

Any c_0 which satisfies

$$\frac{e^{-c_0{}^2/2}}{ec_0} \leqslant \frac{\sqrt{2\pi}}{2} \tag{10}$$

(for example, $c_0 = 1/3$) proves the theorem.

Remark. The smallest value of c_0 satisfying (10) is approximately .282. Taken together, Theorems 2 and 3 imply

THEOREM 4. *Every zero, $u + iv$, of $F(w)$ in the upper half-plane satisfies $uv < -\pi$ and $u + v + c_0 > 0$.*

The set of parabolas (8) can be rewritten

$$\left(y + \frac{uv}{2}\right)^2 = v^2 \left(x + \frac{u^2}{4}\right). \tag{11}$$

Each such parabola passes through the origin (in the (x, y)-plane) and has slope v/u there. Since $u + v + c_0 > 0$ and $uv < -\pi$, we have

$$\frac{v}{u} < -\frac{d^2}{\pi}$$

where

$$d = \frac{(c_0^2 + 4\pi)^{1/2} - c_0}{2}.$$

Finally, then, there exists N such that, for $n > N$, $S_n(z)$ has no zero in the parabolic domain

$$|y| \leqslant -\frac{\pi}{2} + d\left(x + \frac{\pi^2}{4d^2}\right)^{1/2}, \qquad x \geqslant 0. \tag{12}$$

Moreover, if the zero, $u + iv$ of $F(w)$ in the upper half-plane which minimizes v/u is $u_0 + iv_0$ and $u_0 v_0 = -a$ then the parabolic arc

$$y = -\frac{a}{2} + v_0\left(x + \frac{a^2}{4v_0^2}\right)^{1/2}, \qquad x \geqslant 0$$

contains a limit point of zeros $S_n(z)$ as $n \to \infty$.

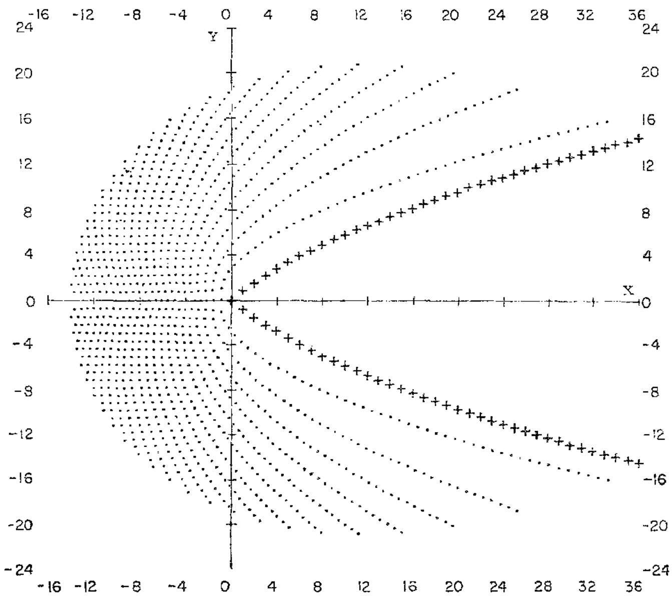

FIG. 1. Zeros of $S_n(z)$, $n = 0, \ldots, 47$ and zero-free parabolic domain.

The zeros of $S_n(z)$ for $n = 0, 1, \ldots, 47$ are shown as dots in Fig. 1, where the domain (12) for $d(.282) = .164$ is outlined by crosses. These calculations were carried out for us with the IBM Fortran Scientific Subroutine Package, on the 360/91, by H. Lewitan.

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